# Ground State Energy Calculations of the $\nu = 1/2$ and the $\nu = 5/2$ FQHE System

J. Dietel

*Institut für Theoretische Physik, Universität Leipzig,*
*Augustusplatz 10, D 04109 Leipzig, Germany*
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We reconsider energy calculations of the spin polarized $\nu = 1/2$ Chern-Simons theory. We show that one has to be careful in the definition of the Chern-Simons path integral in order to avoid an IR divergent magnetic ground state energy in RPA as in[15]. We correct the path integral and get a well behaved magnetic energy by considering the energy of the maximal divergent graphs as well as the Hartree-Fock graphs. Furthermore, we consider the $\nu = 1/2$ and the $\nu = 5/2$ system with spin degrees of freedom. In doing this we formulate a Chern-Simons theory of the $\nu = 5/2$ system by transforming the interaction operator to the next lower Landau level. We calculate the Coulomb energy of the spin polarized as well as the spin unpolarized $\nu = 1/2$ and the $\nu = 5/2$ system as a function of the interaction strength in RPA. These energies are in good agreement with numerical simulations of interacting electrons in the first as well as in the second Landau level. Furthermore, we calculate the compressibility, the effective mass and the excitations of the spin polarized $\nu = 2 + 1/\tilde{\phi}$ systems where $\tilde{\phi}$ is an even number.

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## I. INTRODUCTION

The combination of an electronic interaction and a strong magnetic field in a two-dimensional electron system yields a rich variety of phases. These are best classified by the filling factor $\nu$, which is the electron density divided by the density of a completely filled Landau level. In this work we mainly consider energy calculations on systems with filling factors $\nu = 1/2$ and $\nu = 5/2$. These system are most suitably described by the Chern-Simons theory. In solid state physics this theory is mainly used in the fields of the fractional quantum Hall effect and high temperature superconductivity. The applications of the Chern-Simons theory in the field of high-temperature superconductivity is based on a work of Polyakov[1]. Since the discovery of the fractional quantum Hall effect by Tsui, Störmer and Gossard (1982)[2] there were many attempts to explain this experimental observation. The current Chern-Simons type theories of this effect are mainly based on a work of Jain (1989)[3]. In his theory he mapped the wave functions of the integer quantum Hall effect to wave functions of the fractional quantum Hall effect. In the case of filling fraction $\nu = 1/2$ every electron gets two magnetic flux quantums through this mapping. By this transformation one gets new quasi-particles (composite fermions) which do not see any magnetic field in first approximation (mean field). A field theoretical language for this scenario was first established by Halperin, Lee, Read (HLR) (1992)[4] as well as Kalmeyer and Zhang (1992)[5] for the $\nu = 1/2$ system. The interpretation of many experiments supports this composite fermion picture. We mention transport experiments with quantum (anti-) dots[6], and focusing experiments[7] here. An overview of further experiments can be found in[8].

HLR studied many physical quantities within the random-phase approximation (RPA). Most prominent among these is the effective mass of the composite fermions which they found to diverge at the Fermi surface[4,9]. Recently, Shankar and Murthy[10] proposed a new theory of the $\nu = 1/2$ system. Based upon a transformation of the Chern-Simons Hamiltonian one achieves a separation of the magneto-plasmon oscillators from the total interaction of the system. After restricting the number of the magneto-plasmon oscillators to the number of electrons they got a finite quasi-particle mass which scales with the inverse of the strength of the Coulomb repulsion. In this derivation they calculate a smaller number of self energy Feynman graphs than in the RPA. Just recently Stern et al.[11] calculated the self energy Feynman graphs for the theory of Shankar and Murthy in RPA and got the same divergence of the effective mass as HLR. Besides the theory of Shankar and Murthy there are other alternate formulations of the composite fermionic picture which are formulated entirely in the lowest Landau Level[12−14].

In the following we will use mainly the Chern-Simons theory of HLR to calculate ground state energies of the $\nu = 1/2$ and the $\nu = 5/2$ system. To get composite fermions one has to transform the electronic wave function with a rather singular transformation. Due to this transformation one gets a density-density interaction of the form $\sim 1/k^2$ for small wave vectors. Because of this singular interaction the singular diagrams in leading order should be resummed. These diagrams are well known as the RPA in the jellium problem in $d = 3$. We have shown in an earlier publication[15] that despite of resumming these singular diagrams one gets an IR-divergence in the canonical potential of the $\nu = 1/2$ system in RPA due to the $\sim 1/k^2$ interaction. Thereby we used the Chern-Simons path integral of HLR for the resummation of the diagrams. We obtained further that the Coulomb

part of the energy is finite and in good agreement with the Coulomb energy of numerical simulations of interacting electrons in the lowest Landau level by Morf and d'Ambrumenil[20] as well as Girlich[21]. We have to mention that C. Conti and T. Chakraborty[16] got an excellent agreement with the numerical results by calculating the Coulomb energy with the help of the STLS theory[17] which is a generalisation of the RPA theory. In the following it is shown that we obtained an IR divergence of the magnetic energy in our earlier work because the normal ordering of the Chern-Simons Hamiltonian was not properly taken into account in the path integral of HLR. In[18] we correct this error with the help of an intermediate time technique in the path integral and a change of the coupling between the fermion fields and the bosonic fields. One can show that this path integral has a finite energy in the RPA. Details of the analysis of the RPA for this path integral will be published elsewhere. In the case of the Coulomb gas, the Hartree-Fock graphs belong to the maximal divergent Feynman graphs[19]. This is no longer valid for the Chern-Simons theory. For this theory we will get a finite energy for the spin polarized $\nu = 1/2$ system by calculating the energy of all Feynman-graphs which belong either to the Hartree-Fock graphs or to the maximal divergent graphs (for a given number of interaction vertices). With the help of this principle one gets a well-suited finite approximation of the magnetic energy. Furthermore, we get the same Coulomb energy as in[15].

In the second part of this work we will calculate the Coulomb energy of the $\nu = 1/2$ and the $\nu = 5/2$ system including the spin degree of freedom. The $\nu = 5/2$ system is of theoretical interest because it consists of one Landau level filled with spin up and spin down electrons. Like in the $\nu = 1/2$ system the second Landau level is half filled with electrons. So the $\nu = 5/2$ system should have similar physical properties as the $\nu = 1/2$ system. As a matter of fact this is not the case. Eisenstein et al.[22] were able to show with the help of tilted field experiments that the ground state of the $\nu = 5/2$ system is spin unpolarized and incompressible. This is different to the ground state of the $\nu = 1/2$ system, which is spin polarized and compressible. Numerical simulations of the ground state of the $\nu = 1/2$ and the $\nu = 5/2$ system[23,24] show that one gets a transition from a spin unpolarized ground state to a spin polarized ground state depending on the strength of the interaction between the electrons. For calculating this transition with perturbational methods we will construct a Chern-Simons theory of the $\nu = 2 + 1/\tilde{\phi}$ systems where $\tilde{\phi}$ is an even number by transforming the Coulomb interaction of the second Landau level to the first Landau level. By neglecting the lowest Landau level, the $\nu = 5/2$ system behaves like a $\nu = 1/2$ system with an altered interaction potential. Because of this we can calculate the Coulomb part of the energy of the $\nu = 5/2$ system within the RPA formalism of the $\nu = 1/2$ system. It will be shown that one gets a transition from a spin polarized to a spin unpolarized ground state for the $\nu = 1/2$ system as well

as for the $\nu = 5/2$ system depending on the interaction strength. This transition is in qualitative agreement with the numerical simulations[23,24]. Furthermore, we will use the Chern-Simons theory of the $\nu = 2 + 1/\tilde{\phi}$ systems to calculate the compressibility, the effective mass and the excitations.

As mentioned above we will treat the energy problem of the $\nu = 1/2$ and the $\nu = 5/2$ Chern-Simons system in RPA. The use of this approximation is motivated by the similarity of the interaction potentials of these two systems to the one of the $d = 3$ jellium problem. It is well known that for the latter a perturbational calculation of the ground state energy results in a good approximation especially for small densities (which is a small parameter in this theory). Such a small parameter is not existent in the $\nu = 1/2$ Chern-Simons theory. Until now it is not clear wether the RPA calculations of the ground state energy in Chern-Simons theories are in agreement with the experiments. One aim of this paper is to make one step further to a positive answer to this question.

The paper is organized as follows: In section II we reconsider the polarized $\nu = 1/2$ system with the help of a path integral which takes the normal ordering of the Chern-Simons Hamiltonian into account. We calculate the energy of the maximal divergent graphs together with the Hartree-Fock Feynman graphs of the Chern-Simons Hamiltonian. In section III we formulate the RPA theory of the $\nu = 1/2$ Chern-Simons theory subjected to a spin constraint. In section IV we formulate a Chern-Simons theory of the $\nu = 2 + 1/\tilde{\phi}$ systems and calculate the compressibility, the effective mass and the excitations in RPA. In section V we calculate the Coulomb ground state energy of the spin polarized as well as the spin unpolarized $\nu = 1/2$ and $\nu = 5/2$ systems.

## II. THE GROUND STATE ENERGY OF THE SPIN POLARIZED $\nu = 1/2$ SYSTEM

In this section we consider interacting spin polarized electrons moving in two dimension in a strong magnetic field $B$ directed in the positive $z$-direction of the system. The electronic density of the system is chosen such that the lowest Landau level of a non-interacting system is filled to a fraction $\nu = 1/\tilde{\phi}$ where $\tilde{\phi}$ is an even number. We are mainly interested in $\tilde{\phi} = 2$. After performing a Chern-Simons transformation[26] of the electronic wave function one gets the Hamiltonian of the composite fermions as:

$$H_{CS} = \int d^2 r \left[ \frac{1}{2m} \left| \left( -i\vec{\nabla} + \vec{A} + \vec{a}_{CS} \right) \Psi(\vec{r}) \right|^2 \right. \tag{1}$$

$$\left. + \frac{1}{2} \int d^2 r' \left\{ (|\Psi(\vec{r})|^2 - \rho_B) V^{ee}(|\vec{r} - \vec{r}'|)(|\Psi(\vec{r}')|^2 - \rho_B) \right\} \right].$$

The Chern-Simons operator $\vec{a}_{CS}$ is defined by $\vec{a}_{CS}(\vec{r}) = \tilde{\phi} \int d^2 r' \ \vec{f}(\vec{r} - \vec{r}') \Psi^+(\vec{r}') \Psi(\vec{r}')$. Here $\Psi^+(\vec{r})$ creates (and

$\Psi(\vec{r})$ annihilates) a composite fermion with coordinate $\vec{r}$. $V^{ee}(r) = e^2/r$ is the Coulomb interaction where $e^2 = q_e^2/\epsilon$. $q_e$ is the charge of the electrons and $\epsilon$ is the dielectric constant of the background field $\rho_B$. $\vec{A}(\vec{r})$ is the vector potential $\vec{A} = 1/2\,\vec{B} \times \vec{r}$ and $\vec{B}$ is a homogeneous magnetic field in z-direction $\vec{B} = B\vec{e}_z$ where $\vec{e}_z$ is the unit vector in $z$-direction. We suppose throughout this paper that $B$ is a positive number. The function $\vec{f}(\vec{r})$ is given by $\vec{f}(\vec{r}) = -\vec{e}_z \times \vec{r}/r^2$. We used the convention $\hbar = 1$ and $c = 1$ in the above formula (1). Furthermore, we set $q_e = 1$ for the coupling of the magnetic potential to the electrons. It is well known, that the partition function of the Hamiltonian (1) can be written in an operator formalism[1] with the help of the bosonic Chern-Simons fields $(a^0(r,t), \vec{a}(r,t))$. This is shown in the first quantized path integral language. Using standard methods[27] we can transform this partition function to a path integral. This path integral is written as

$$Z_{1/2} = \lim_{\epsilon \to 0} \frac{1}{\overline{N}_{1/2}} \prod_{l=1}^{N_l} \int_{\mathrm{BC}} \mathcal{D}[a_l^0, \vec{a}_l, \sigma_l]\, \mathcal{D}[\Psi_l^*, \Psi_l] \qquad (2)$$
$$\times \exp\left[-\epsilon\left(L_l + L_{CS,l} + L_{ee,l} + L_{ee,l}^0\right)\right] .$$

The various functions in (2) are given by

$$L_l = \int d^2r\, \Psi_l^*(\vec{r}) \frac{1}{\epsilon}\left(\Psi_l(\vec{r}) - \Psi_{l-1}(\vec{r})\right) \qquad (3)$$
$$-\Psi_l^*(\vec{r})\left(\mu + \left(1 + \frac{\epsilon}{2}a_l^0(\vec{r})\right)a_l^0(\vec{r})\right)\Psi_{l-1}(\vec{r})$$
$$+\frac{1}{2m}\Psi_l^*(\vec{r})\left(-i\vec{\nabla} + \vec{A}(\vec{r}) + \vec{a}_l(\vec{r})\right)^2 \Psi_{l-1}(\vec{r}) ,$$

$$L_{CS,l} = \frac{1}{2\pi\tilde{\phi}} \int d^2r\, a_l^0(\vec{r})\, \vec{\nabla} \times \vec{a}_l(\vec{r}) , \qquad (4)$$

$$L_{ee,l} = \int d^2r\, d^2r'\sigma_l(\vec{r})\, V^{ee}(|\vec{r} - \vec{r}'|)$$
$$\times\left(\Psi_l^*(\vec{r}')\Psi_{l-1}(\vec{r}') - \rho_B\right) , \qquad (5)$$

$$L_{ee,l}^0 = -\frac{1}{2}\int d^2r\, d^2r'\, \sigma_l(\vec{r})\, V^{ee}(|\vec{r} - \vec{r}'|)\, \sigma_l(\vec{r}') , \qquad (6)$$

together with the norm

$$\overline{N}_{1/2} = \prod_{l=1}^{N_l} \int_{\mathrm{BC}} \mathcal{D}[a_l^0, \vec{a}_l, \sigma_l]\exp\left[-\epsilon\left(L_{CS,l} + L_{ee,l}^0\right)\right] . \qquad (7)$$

The path integral (2) is correct under the gauge condition $\vec{\nabla} \cdot \vec{a}_l = 0$. $a_l^0$ is a imaginary field. $\vec{a}_l$ and $\sigma_l$ are real fields. The time slices $\epsilon$ are defined as $\epsilon = \beta/N_l$. The index $l$ counts the discrete time slices. Furthermore, we have anti-periodic boundary conditions $\Psi_{N_l} = -\Psi_0$ for the Grassmann fields[27]. The action of the path integral (2) is given by a fermionic term $L_l$, a bosonic term $L_{CS}$ of the Chern-Simons form, and two Coulomb interaction terms $L_{ee,l}$, $L_{ee,l}^0$. In comparison to the Chern-Simons path integral of HLR[4] which was used in our

earlier publication[15] we get an additional term proportional to $\epsilon a_l^0(\vec{r})^2$ in $L_l$ (3). This term is mainly due to the normal-ordering of the $\Psi^6$ term in the Chern-Simons Hamiltonian $H_{CS}$ (1). This is most easily seen by integrating the path integral (2) over the Chern-Simons fields. Due to the additive term one can not perform the formal limit $\epsilon \to 0$ in (2). This limit has to be taken in every additive term after integration over the Chern-Simons and fermionic fields. Furthermore, we have to remark, that we treat the same formalism as above in order to derive the path integral of Shankar and Murthy[10,11]. There we get no additional term in comparison to their path integral[28].

In the following we integrate (2) over the fermionic fields. Using a mean field expansion of the result up to the second order in the bosonic fields one gets after integration for the grand-canonical potential $\Omega = \Omega_0 + \Omega_{\mathrm{RPA}}$. $\Omega_0$ is the grand-canonical potential of the Coulomb-free electron gas. We now split $\Omega_{\mathrm{RPA}}$ into a variety of terms for which one can take the limit $\epsilon \to 0$ rather easily[15].

$$\Omega_{\mathrm{RPA}} = \Omega_R + \mathrm{F}^{ee} + \mathrm{F}^{CS} + \mathrm{H} . \qquad (8)$$

The expression $\Omega_R$ corresponds to the RPA graphs which do not belong to the Hartree-Fock graphs. $\mathrm{F}^{ee} + \mathrm{F}^{CS}$ are the Coulomb and Chern-Simons Fock diagrams. H are the Hartree diagrams. We will see below that the Hartree term is due to the normal ordering of the $\Psi^6$-term in (1). For temperature $T = 0$, $\Omega_R$ is given by

$$\Omega_R = \frac{1}{2(2\pi)^3} \int d\omega\, d^2q\left[\log\left(1 + \mathrm{CS}(q,\omega) + \mathrm{EM}(q,\omega)\right)\right.$$
$$\left. -\mathrm{CS}_1(q,\omega) - \mathrm{EM}(q,\omega)\right] . \quad (9)$$

The functions CS, $\mathrm{CS}_1$ and $EM$ are defined as

$$\mathrm{CS} = -(2\pi\tilde{\phi})^2 \frac{1}{q^2}\Pi_{a^0a^0}\left(\Pi_{aa} + \frac{\mu}{2\pi}\right) , \qquad (10)$$

$$\mathrm{CS}_1 = -(2\pi\tilde{\phi})^2 \frac{1}{q^2}\Pi_{a^0a^0}\frac{\mu}{2\pi} , \qquad (11)$$

$$\mathrm{EM} = -e^2\frac{2\pi}{q}\Pi_{a^0a^0} . \qquad (12)$$

$EM$ is the Coulomb term and $CS$ is the Chern-Simons term of the RPA formula. The second and third summands $EM$ and $CS_1$ in (9) are given by the first order graphs of the RPA which have to be subtracted for treating equal time Green's function in the right way[15]. The density-density response function $\Pi_{a^0a^0}(q,\omega,\mu)$ and the transversal current-current response function $\Pi_{aa}(q,\omega,\mu)$ are given in (A2) of appendix A. The Hartree-Fock terms of $\Omega_{\mathrm{RPA}}$ are given by

$$\mathrm{H} = \frac{1}{2}(2\pi\tilde{\phi})^2\frac{m\mu^2}{(2\pi)^4}\int d^2q\,\frac{1}{q^2} , \qquad (13)$$

$$\mathrm{F}^{XX} = \frac{1}{2(2\pi)^3}, \qquad (14)$$

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$$\times \int d^2k\, d^2q\; n_F(k,\mu)\, n_F(q,\mu)\, V^{XX}(|\vec{k}-\vec{q}|)$$

The function $n_F(q,\mu)$ is the fermion occupation factor for the chemical potential $\mu$. The vertices $V^{XX}$ are defined by

$$V^{ee}(q) = e^2 \frac{1}{q} \quad , \qquad V^{CS}(q) = \tilde{\phi}^2 \mu \frac{1}{q^2}\,. \qquad (15)$$

Here $V^{ee}$ is the Coulomb vertex. $V^{CS}$ is the Chern-Simons vertex $\sim 1/q^2$ mentioned in the introduction. Comparing $\Omega_{\mathrm{RPA}}$ with the analogous expression in[15] the only difference between the two expressions is the Hartree term H in (13), which corresponds to a Hartree Feynman graph of $H_{CS}$ (1) resulting from the normal ordering of the $\Psi^6$ term. It is shown in[15], that $\Omega_{\mathrm{RPA}} - \mathrm{NO}$ is IR-divergent. This is based on the IR-divergence in $\mathrm{F}^{CS}$. The additive term H shifts this IR-divergent grand canonical potential in[15] to an UV-divergent grand canonical potential $\Omega_{\mathrm{RPA}}$. As mentioned above $\mathrm{F}^{CS}$ and H belong to the Hartree-Fock Feynman graphs of $H_{CS}$. Therefore, if we calculate the energy of all Hartree-Fock Feynman graphs in addition to $\Omega_R$ we will get a finite energy because the Hartree-Fock energy of $H_{CS}$ (1) has to be finite. Furthermore, it is easily seen that $\Omega_R$ corresponds to the maximal divergent graphs for a given number of vertices. To get a good approximation of the energy we take into account in the following the maximal divergent graphs (per number of vertices) together with the Hartree-Fock graphs. As mentioned in the introduction this energy principle is also used for a calculation of the ground state energy of the Coulomb gas by many authors (e.g.[19]). In order to calculate that energy we have to determine at first the Hartree-Fock energy of $H_{CS}$. Sitko calculated in[29] the Hartree-Fock energy of the magnetic part of $H_{CS}$. The Coulomb part of the Hartree-Fock energy consists only of $\mathrm{F}^{ee}$ (14). This integral is easily evaluated yielding

$$U_{HF} = \frac{m}{4\pi}\mu^2 + \frac{3m}{16\pi}\tilde{\phi}^2\mu^2 - \frac{2\sqrt{2}}{3\pi^2}e^2 m^{\frac{3}{2}}\mu^{\frac{3}{2}} \qquad (16)$$

for the Hartree-Fock energy of the Chern-Simons Hamiltonian $H_{CS}$. The first term in (16) is the kinetic energy of the composite fermions. For the $\nu = 1/2$ system we have to insert $\tilde{\phi} = 2$ into $U_{HF}$. Doing this we obtain two times the value of the exact magnetic energy for $U_{HF}$ ($\mu = (2\pi)\rho/m$).
With the help of an $e^2$-expansion of $\Omega_R$ in (9) we can calculate the coefficients of the expansion for the $\nu = 1/2$ system numerically:

$$\Omega_R = -0.19\, m\,\mu^2 - 0.038\, e^2 m^{\frac{3}{2}}\mu^{\frac{3}{2}} + O(e^4)\,. \qquad (17)$$

The elctron density $\rho$ is given by $\mu = \rho/(2\pi m)$. This can be motivated by the Luttinger-Ward theorem[30] under the consideration that there are no anomalous graphs

in our approximation of the energy. Thus we get from (16), (17)

$$U^{\mathrm{Th}} \approx 5.06\frac{\rho^2}{m} - 2.11e^2\rho^{\frac{3}{2}}\,,\; U^{\mathrm{Num}} \approx 6.28\frac{\rho^2}{m} - 1.67e^2\rho^{\frac{3}{2}}$$

for the energy density $U^{\mathrm{Th}} = U_{HF} + \Omega_R$ of the $\nu = 1/2$ system. It is evident by the selection principle of the calculated Feynman graphs that the Coulomb energy part in $U^{\mathrm{Th}}$ is the same as in[15]. To compare our perturbative result for the energy with the energy obtained by other methods we have written down in $U^{\mathrm{Num}}$ the exact magnetic energy and the Coulomb energy obtained by numerical diagonalization techniques. This Coulomb energy was calculated by Morf and d'Ambrumenil[20] as well as Girlich[21] by diagonalization of the Coulomb part of the Hamiltonian for electrons on a sphere in the lowest Landau level. One sees that perturbational and the numerical calculated energies are in satisfactory agreement.

## III. THE COULOMB ENERGY OF THE $\nu = 1/2$ SYSTEM INCLUDING CONSTRAINTS ON THE TOTAL SPIN

In the $\nu = 1/2$ as in the $\nu = 5/2$ system appears an interesting phase transition from a spin unpolarized ground state to a spin polarized ground state by modifying the interaction strength between the electrons. In the next few sections we will see, that this physical phenomenon already exists on the level of the RPA of the Chern-Simons theory. To show this we have to add a spin constraint variable into the Chern-Simons path integral. Without this spin-constraint variable the $\nu = 1/2$ Chern-Simons path integral is written as in (2) with a doubling of the fermionic Grassmann fields representing the spin-up and spin-down freedom of the electrons. In the following we denote by $c_\uparrow^+(\vec{r})$, $(c_\downarrow^+(\vec{r}))$ the creation operator of one composite fermion at point $\vec{r}$ with spin up (spin down). $c_\uparrow(\vec{r})$, $(c_\downarrow(\vec{r}))$ annihilates one composite fermion with spin up (spin down) at point $\vec{r}$.
The total spin in the xy-plane is given by

$$S_x^2 + S_y^2 = \int d^2r \left( \frac{1}{2}c_\uparrow^+(\vec{r})\, c_\uparrow(\vec{r}) + \frac{1}{2}c_\downarrow^+(\vec{r})\, c_\downarrow(\vec{r}) \right) + S_{\mathrm{Mod}}^2,$$

$$S_{\mathrm{Mod}}^2 = - \int d^2r\, d^2r'\; c_\uparrow^+(\vec{r})\, c_\downarrow^+(\vec{r}')\, c_\downarrow(\vec{r})\, c_\uparrow(\vec{r}')\,. \qquad (18)$$

Furthermore, we define from $H_{CS}$ (1) a new spin constraint Chern-Simons Hamiltonian $H_{CS}^s$ by $H_{CS}^s = H_{CS} - \mu_s S_{\mathrm{Mod}}^2$. We will fix the total spin $S^2 = S_x^2 + S_y^2 + S_z^2$ by the constraint such that $S^2$ is minimal for a given $S_z$. This is the case if $S^2 = |S_z|(|S_z| + 1)$. One can get this constraint on $S^2$ by differentiating $-1/\beta \log(Z^s)$ with respect to $\mu_s$. Thereby we denote the partition function of $H_{CS}^s$ by $Z^s$. Thus we get the following conditions on the partition function $Z^s$

$$-\frac{1}{\beta}\left( \frac{\partial}{\partial\mu_\uparrow} + \frac{\partial}{\partial\mu_\downarrow} \right) \log(Z^s) = \langle|N|\rangle$$

$$-\frac{1}{2}\frac{1}{\beta}\left(\frac{\partial}{\partial\mu_\uparrow}-\frac{\partial}{\partial\mu_\downarrow}\right)\log(Z^s)=\langle|S_z|\rangle\ ,$$

$$\frac{1}{\beta}\frac{\partial}{\partial\mu_s}\log(Z^s)=-\frac{1}{4}\frac{1}{\beta^2}\left(\frac{\partial}{\partial\mu_\uparrow}-\frac{\partial}{\partial\mu_\downarrow}\right)^2\log(Z^s)$$
$$-\frac{1}{2}\langle|N|\rangle+|\langle|S_z|\rangle|\ . \quad (19)$$

The symbol $\langle|\cdot|\rangle$ is the expectation value in the Gibb's state of the Hamiltonian $H^s_{CS}$. $N$ is the particle number operator $N=\int d^2r\,c^+_\uparrow(\vec{r})c_\uparrow(\vec{r})+c^+_\downarrow(\vec{r})c_\downarrow(\vec{r})$. $\mu_\uparrow$ and $\mu_\downarrow$ are the chemical potentials of the system representing the spin-up and spin-down electrons. Furthermore, we have to mention that one has to be careful in using equation (19). This is because the first term on the right hand side of equation (19) equals to $\langle|S_z^2|\rangle-\langle|S_z|\rangle^2$ for a finite system. This is no longer the case for an infinite system (an example is the kinetic Hamiltonian). Thus one has to differentiate first for a finite system and afterwards take the limit to an infinite system. We are now able to write down the $\nu=1/2$ Chern-Simons path integral. Using the Hubbard-Stratonovich decoupling of the fermionic fields in the operator $S^2_{\mathrm{Mod}}$ we get

$$Z^s_{1/2}=\lim_{\epsilon\to0}\frac{1}{\overline{N}^s_{1/2}}\prod_{l=1}^{N_l}\int_{\mathrm{BC}}\mathcal{D}[a^0_l,\vec{a}_l,\sigma_l,\sigma^s_l]\mathcal{D}[c^*_{\kappa,l},c_{\kappa,l}]\quad(20)$$
$$\times\exp\left[-\epsilon\left(L^s_l+L_{s,l}+L_{ee,l}+L^0_{ee,l}+L_{CS,l}+L^0_{s,l}\right)\right]$$

with the help of

$$L^s_l=\sum_{\kappa\in\{\uparrow,\downarrow\}}\int d^2r\,c^*_{\kappa,l}(\vec{r})\frac{1}{\epsilon}\left(c_{\kappa,l}(\vec{r})-c_{\kappa,l-1}(\vec{r})\right)$$
$$-c^*_{\kappa,l}\left(\mu_\kappa+\left(1+\frac{\epsilon}{2}a^0_l(\vec{r})\right)a^0_l(\vec{r})\right)c_{\kappa,l-1}(\vec{r})$$
$$+c^*_{\kappa,l}(\vec{r})\frac{1}{2m}\left(-i\vec{\nabla}+\vec{A}(\vec{r})+\vec{a}_l(\vec{r})\right)^2c_{\kappa,l-1}(\vec{r}),\quad(21)$$

$$L_{ee,l}=\int d^2r\,d^2r'\,\sigma_l(\vec{r})\,V^{ee}(|\vec{r}-\vec{r}'|)$$
$$\times\left(\sum_{\kappa\in\{\uparrow,\downarrow\}}c^*_{\kappa,l}(\vec{r}')c_{\kappa,l-1}(\vec{r}')-\rho_B\right),\quad(22)$$

$$L_{s,l}=\mu_s\int d^2r\,d^2r'\,\sigma^s_l(\vec{r})\left(c^*_{\uparrow,l}(\vec{r}')\,c_{\downarrow,l-1}(\vec{r}')\right)\ ,\quad(23)$$

$$L^0_{ee,l}=-\frac{1}{2}\int d^2r\,d^2r'\,\sigma_l(\vec{r})\,V^{ee}(|\vec{r}-\vec{r}'|)\,\sigma_l(\vec{r}')\ ,\quad(24)$$

$$L^0_{s,l}=-\frac{\mu_s}{4}\int d^2r\,d^2r'\,(\sigma^s_l(\vec{r})\sigma^s_l(\vec{r}'))\quad(25)$$

and the normalizing factor

$$\overline{N}^s_{1/2}=\prod_{l=1}^{N_l}\int_{\mathrm{BC}}\mathcal{D}[a^0_l,\vec{a}_l,\sigma_l,\sigma^s_l]$$
$$\times\exp\left[-\epsilon\left(L_{CS,l}+L^0_{ee,l}+L^0_{s,l}\right)\right]\ .\quad(26)$$

The terms $L_{s,l}$, $L^0_{s,l}$ in (20) are given by the Hubbard-Stratonovich decoupling of the spin term $S^2_{\mathrm{Mod}}$. After

integrating in (20) over the fermionic fields we get the following mean-field equations for the bosonic fields

$$\vec{\nabla}\times\vec{a}_l=(2\pi\tilde{\phi})\,B\quad,\quad a^0_l=0\ ,$$
$$\sigma_l=0\quad,\quad \sigma^s_l=0\ .\quad(27)$$

The mean-field Green's function is given by $G^s(q,\omega)=(-1/(i\omega-q^2/(2m)+\mu_\uparrow),-1/(i\omega-q^2/(2m)+\mu_\downarrow))$. Integrating this mean-field expansion to second order of the bosonic fields we get for the grand canonical potential $\Omega^s_{1/2}=\Omega^s_0+\Omega^s_{\mathrm{RPA}}$. $\Omega^s_0$ is the grand canonical potential for a spin dependent electron gas without interaction. As we do not have any anomalous graphs, we want to make use of the Luttinger-Ward theorem[30]. This theorem states that a good approximation is obtained for $\mu_\uparrow$, $\mu_\downarrow$ if one uses $m(\mu_\uparrow+\mu_\downarrow)/(2\pi)=\rho$ and $m(\mu_\uparrow-\mu_\downarrow)/(4\pi)=\langle|S_z|\rangle/F$. $F$ is the area of the system. As a function of $\mu_\uparrow$, $\mu_\downarrow$, the expectation value $\langle|S_z|\rangle$ behaves like a step function for a finite system. Because the first term on the right hand side of equation (19) is $-1/(2\beta)(\partial/\partial\mu_\uparrow-\partial/\partial\mu_\downarrow)\langle|S_z|\rangle$ we observe that this term is zero. Because $\Omega_0$ does not depend on $\mu_s$, it can not fulfill the spin constraint equation (19). On the other hand it is clear that the ground state of the mean-field Hamiltonian fulfills the spin constraint $\langle|S^2|\rangle=|\langle S_z\rangle|(|\langle S_z\rangle|+1)$. For this reason $\Omega_0$ together with the grand canonical potential of the exchange graph of $S^2_{\mathrm{Mod}}$ (18) fulfills the spin constraint equation (19) (The exchange graph is a part of $\Omega^s_{\mathrm{RPA}}$). Furthermore, we have to mention that the exchange part of $S^2_{\mathrm{Mod}}$ is the only part of $\Omega^s_{\mathrm{RPA}}$ which is linear in $\mu_s$. Thus the spin constraint equation (19) is fulfilled for $\mu_s=0$. In the following we denote $\Pi(q,\omega,\mu_\uparrow)+\Pi(q,\omega,\mu_\downarrow)$ by $\Pi^d(q,\omega,\mu_\uparrow,\mu_\downarrow)$. $\Pi$ are the diverse response functions of appendix A. After some transformations and an expansion in $e^2$ we get with the help of the functions

$$\mathrm{CS}^s=-(2\pi\tilde{\phi})^2\frac{1}{q^2}\Pi^d_{a^0a^0}\left(\Pi^d_{aa}+\frac{\mu_\uparrow+\mu_\downarrow}{2\pi}\right),\quad(28)$$

$$\mathrm{EM}_{1/2}=-(2\pi)V^{ee}(q)\,\Pi^d_{a^0a^0}\ ,\quad(29)$$

$$\mathrm{F}^{ee}_{1/2}=-\sum_{\kappa\in\{\uparrow,\downarrow\}}\frac{1}{2(2\pi)^3}\quad(30)$$
$$\times\int d^2k\,d^2q\,V^{ee}(|\vec{k}-\vec{q}|)n_F(k,\mu_\kappa)n_F(q,\mu_\kappa)$$

the $e^2$-part of the grand-canonical $\Omega^s_{1/2}$ as

$$\Omega^{ee}_{1/2}=\mathrm{F}^{ee}_{1/2}\quad(31)$$
$$+\frac{1}{2(2\pi)^3}\int d\omega\,d^2q\left(\frac{1}{(1+\mathrm{CS}^s(q,\omega))}-1\right)\mathrm{EM}_{1/2}(q,\omega).$$

After developing the Chern-Simons theory of the $\nu=5/2$ system in the next section, we will calculate $\Omega^{ee}_{1/2}$ in section V numerically.

## IV. THE CHERN-SIMONS THEORY OF THE $\nu = 5/2$ SYSTEM

The $\nu = 5/2$ system consists of one lowest Landau level filled by spin up and spin down electrons. The second Landau level is half filled. After performing the Chern-Simons transformation of the Hamiltonian of this system one does not get a mean field free theory as in the case of the $\nu = 1/2$ system. It is clear that the filled lowest Landau level of the $\nu = 5/2$ system should not have much relevance for the physical properties. Thus we will consider the $\nu = 5/2$ system as a $\nu = 1/2$ system with a modified interaction potential between the electrons. If we define an isometric transformation which transforms a wave function of a higher Landau level to the next lower Landau level then the calculation of the Coulomb energy in the second Landau level corresponds to the calculation of the energy of the first Landau level with the transformed interaction potential of the electrons. If one calculates the Coulomb energy by perturbational methods then this transformation should not change the wave function too much. We will develop in the following an isometric transformation which leaves the center of mass of the wave function invariant.

The ladder operators between Landau levels are defined with the help of the operators $\Pi_L^x = -i\partial_x + A^x$ and $\Pi_L^y = -i\partial_y + A^y$ by

$$\Pi_L = \Pi_L^x - i\Pi_L^y \quad , \qquad \Pi_L^+ = \Pi_L^x + i\Pi_L^y . \quad (32)$$

The ladder operators act on a wave function belonging to the n-th Landau level as follows

$$\Pi_L^+|n,q> = \sqrt{2B(n+1)}\,|n+1,q>,$$
$$\Pi_L|n,q> = \sqrt{2Bn}\,|n-1,q> \quad (33)$$

With the help of some commutation relations[31] one can easily see that

$$\frac{1}{2B(n+1)}< n,q|\Pi_L\,x\,\Pi_L^+|n,q> = < n+1,q|x|n+1,q>$$

$$\frac{1}{2Bn}< n,q|\Pi_L^+\,x\,\Pi_L|n,q> = < n-1,q|x|n-1,q> \quad (34)$$

The same relation holds for the $y$-coordinate. The partial isometric transformation $P$ which is descending the Landau level functions is given by

$$P|n,q> := |n-1,q> . \quad (35)$$

With the help of the ladder operators (32) the operator $P$ is written by

$$P = \frac{\Pi_L}{(2B)^{\frac{1}{2}}} + \Pi_L^+\Pi_L^2\,\frac{1}{(2B)^{\frac{3}{2}}\sqrt{2}}\left(1 - \sqrt{2}\right)$$
$$+ \Pi_L^{+2}\Pi_L^3\,\frac{1}{(2B)^{\frac{5}{2}}\,2\sqrt{3}}\left(1 + \sqrt{3}\left(1 - \sqrt{2}\right)\right)$$
$$+ \Pi_L^{+3}\Pi_L^4\,O\left(\frac{1}{B^{\frac{7}{2}}}\right) . \quad (36)$$

It is easily seen from equation (36) that the n-th term in $P$ transforms a wave function which belongs to the n-th lowest Landau level to zero. The higher order terms in $P$ are motivated by the different normalization factors of the individual Landau levels in equation (33). We observe from the equations (33) and (34) that the operator $P$ leaves the center of mass coordinate of one Landau level function invariant.

We will use this operator $P$ for transforming the Coulomb interaction to one lower Landau level. Almost all publications carrying out ground state energy calculations of the $\nu = 1/2$ and the $\nu = 5/2$ system by numerical methods do not take into account Landau level splitting[20,24,23]. In these calculations one diagonalizes the Coulomb interaction operator in the second Landau level in the case of the $\nu = 5/2$ system. It is easily seen from perturbation theory that this energy is given by the $e^2$-term of the Coulomb energy. Because we want to compare the energy given by perturbational calculations and numerical calculations we can neglect the higher order $1/B$ terms in $P$ under the condition of calculating the Coulomb energy to order $e^2$. So we get for the isometric partially transformed Coulomb interaction operator

$$\mathcal{V}_{5/2}^{ee} = e^2\,\frac{1}{(2B)^2}\int d^2r\,d^2r'\,\Pi_L\Psi^+(\vec{r})\,\Pi_L\Psi^+(\vec{r}')$$
$$\times \frac{1}{|\vec{r} - \vec{r}'|}\,\Pi_L^+\Psi(\vec{r}')\,\Pi_L^+\Psi(\vec{r}) . \quad (37)$$

$B$ is given by $(2\pi\tilde{\phi})\rho$ where $\rho$ is the density of the electrons in the second Landau level. Since one does not start from a Chern-Simons transformed wave function in the lowest Landau level in the perturbational calculation but from a Slater determinant of plane waves, one should also take into account higher order terms of $P$ in perturbation theory. According to equation (33) these only differ from the first order term by a normalizing factor. Therefore we will fix an effective $B$-field $B_{\text{eff}}$ in equation (37) when proceeding

$$P \approx (1/\sqrt{2B_{\text{eff}}})\,\Pi_L \quad (38)$$

This effective field will be brought into line with property (36). The Chern-Simons transformed operator $P_{CS}$ of $P$ is constructed from $P$ in (36) by the substitution $\vec{A} \to \vec{A} + \vec{a}$. It is approximated through $P_{CS} \approx (1/\sqrt{2B_{\text{eff}}})\,\Pi_{L,CS}$ with the operator $\Pi_{L,CS}$ constructed from $\Pi_L$ by the substitution $\vec{A} \to \vec{A} + \vec{a}$. The $\nu = 5/2$ Chern-Simons path integral $Z_{5/2}^s$ is constructed from the $\nu = 1/2$ Chern-Simons path integral $Z_{1/2}^s$ by the substitution $L_{ee,l} \to L_{ee,l}^{5/2}$. The function $L_{ee,l}^{5/2}$ is given by

$$L_{ee,l}^{5/2} = \int d^2r\,d^2r'\,\sigma_l(\vec{r})\,V^{ee}(|\vec{r} - \vec{r}'|) \quad (39)$$

$$\times \left( \sum_{\kappa \in \{\uparrow, \downarrow\}} P_{CS}\, c^*_{\kappa,l}(\vec{r}')\, P^+_{CS}\, c_{\kappa,l-1}(\vec{r}') - \rho_B \right).$$

The background field $\rho_B$ is given by

$$\rho_B = \sum_{\kappa \in \{\uparrow, \downarrow\}} \langle | \mathcal{P}_{CS}\, c^+_\kappa(\vec{r})\, \mathcal{P}^+_{CS} c_\kappa(\vec{r}) | \rangle \,. \tag{40}$$

$\mathcal{P}_{CS}$ is constructed from $P$ in (36) by the substitution $\vec{A} \to \vec{A} + \vec{a}_{CS}$. Because of this equation the Hartree graphs in the $\nu = 5/2$ Chern-Simons theory are cancelled by the $\rho_B$ couplings. Like in the $\nu = 1/2$ system we are able to calculate the Coulomb energy of the path integral $Z^s_{5/2}$ in RPA. Like in the case of the $\nu = 1/2$ system we obtain $\mu_s = 0$ with the inclusion of the spin constraint (19). We now calculate the Coulomb energy of the path integral (20), (39), with the approximation $P_{CS} \approx (1/\sqrt{2B_{\rm eff}})\, \Pi_{L,CS}$ in RPA. This approximation is also used in (40). For this we define the $\nu = 5/2$ Coulomb potential by $V^{ee}_{5/2}(\vec{r}) = e^2/(2B_{\rm eff} r)$. With the help of the expressions

$$\mathrm{EM}_{5/2} = 2\,(2\pi\tilde{\phi})\frac{2\pi i}{q} V^{ee}_{5/2}(q)\, \Pi^d_{\sigma a^0} \Pi^d_{a\sigma} \tag{41}$$

$$-(2\pi\tilde{\phi})^2 \frac{2\pi}{q^2} V^{ee}_{5/2}(q)\, \Pi^d_{a^0 a^0} \Pi^d_{\sigma a} \Pi^d_{a\sigma}$$

$$-(2\pi\tilde{\phi})^2 \frac{2\pi}{q^2} V^{ee}_{5/2}(q)\, \left( \Pi^d_{aa} + \frac{\mu_\uparrow + \mu_\downarrow}{2\pi} \right) \Pi^d_{\sigma a^0} \Pi^d_{a^0 \sigma} \,,$$

$$\mathrm{F}^{ee}_{5/2} = - \sum_{\kappa \in \{\uparrow, \downarrow\}} \tag{42}$$

$$\frac{1}{2(2\pi)^3} \int d^2k\, d^2q\, k^2 q^2\, V^{ee}_{5/2}(|\vec{k} - \vec{q}|)\, n_F(k, \mu_\kappa)\, n_F(q, \mu_\kappa)$$

we get the RPA Coulomb energy by

$$\Omega^{ee}_{5/2} = \mathrm{F}^{ee}_{5/2} \tag{43}$$

$$+ \frac{1}{2(2\pi)^3} \int d\omega\, d^2q\, \frac{1}{(1 + \mathrm{CS}^s(q,\omega))} \mathrm{EM}_{5/2}(q,\omega) \,.$$

The several response functions $\Pi$ in (41) are defined in (A1) of the appendix. Now we have to fix the effective magnetic field $B_{\rm eff}$ by perturbational methods. We will treat at first the case of a spin polarized ground state (We have to insert $\mu_\downarrow = 0$ and $\Pi(q,\omega,\mu=0) = 0$ in (41) and (42)). For fixing $B_{\rm eff}$ we have look for the main contributions to the momentum integrals of $\Omega^{ee}_{5/2}$ in (43). We will fix $B_{\rm eff}$ so that we get for these momentums the same integrand as we would calculate the path integral (20), (39) with the exact operator $P$ (36) in $L^{5/2}_{ee,l}$. Numerical integration of the two terms in $\Omega^{ee}_{5/2}$ (43) yields results for $\mathrm{F}^{ee}_{5/2}$ which are about a factor 3 smaller compared to the second term. This is in contrast to the $\nu = 1/2$ system. When fixing $1/B^2_{\rm eff}$ in $V^{ee}_{5/2}$ there is no impact on the proportion of the two terms. Thus we use the second term in (43) for fixing $B_{\rm eff}$. After performing the $\omega$ integration in (43) the $q$ integrand of the second term has

the largest contribution to the energy for $q \approx 0$. This is shown in figure 1. Thus the ring momentums $k \approx k_F$ are the most important wave vectors in the calculation of the response functions (A1) in order to calculate the ground state energy of the $\nu = 5/2$ system. This is due to a $n_F(\vec{k} + \vec{q}) - n_F(\vec{k})$ term as a result of integrating (A1) over the ring frequencies. Thus we find the following equation to get the same $q \approx 0$-integrand in the second term in (43) using the exact $P_{CS}$ (36) and its approximation (38) in the path integral

$$\langle u_{\vec{k}_F} | P_{CS} P^+_{CS} | u_{\vec{k}_F} \rangle \Big|_{\vec{a}=\vec{A}} \tag{44}$$

$$= \frac{1}{2B_{\rm eff}} \langle u_{\vec{k}_F} | \Pi_{L,CS} \Pi^+_{L,CS} | u_{\vec{k}_F} \rangle \Big|_{\vec{a}=\vec{A}} = \frac{1}{2B_{\rm eff}} k^2_F \,.$$

$u_{\vec{k}_F}$ is the one particle function $u_{\vec{k}_F} = 1/\sqrt{F} \exp[i\vec{k}_F \vec{r}]$ where $\vec{k}_F$ is a vector of length $k_F = \sqrt{2m\mu}$. Furthermore, we have to insert the mean-field condition $\vec{a} = \vec{A}$ in $P_{CS}$ of equation (44). By using the definitions (35), (36) the left hand side of equation (44) is equal to

$$\langle u_{\vec{k}_F} | P_{CS} P^+_{CS} | u_{\vec{k}_F} \rangle \Big|_{\vec{a}=\vec{A}} \tag{45}$$

$$= \lim_{k \to 0} \langle u_{\vec{k}} | P\, P^+ | u_{\vec{k}} \rangle \Big|_{B_H = k^2\, B_{\rm eff}/k^2_F} = 1 \,.$$

In this equation $P$ is calculated with the vector potential $\vec{A} = B_H/2\,(-y, x)$. The magnetic field $B_H$ is defined by $B_H = k^2\, B_{\rm eff}/k^2_F$.
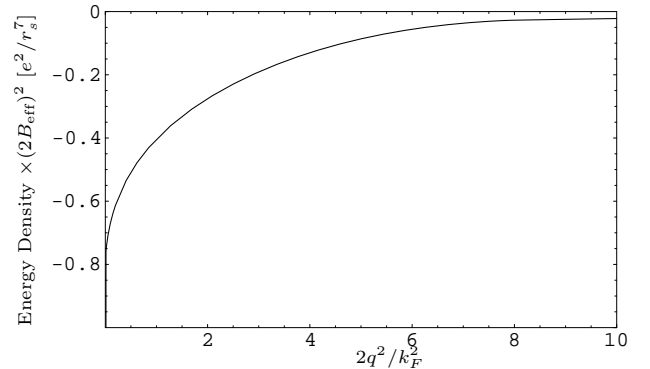


FIG. 1. The contribution of the scaled momentum $2q^2/k^2_F$ to $(2B_{\rm eff})^2$ times the Coulomb energy density for the spin polarized $\nu = 5/2$ system. $(2B_{\rm eff})^2$ times the energy density is shown in units of $e^2/r^7_s$ where $r_s$ is the effective electron distance $\rho = 1/(\pi r_s)^2$ .

From the equations (44) and (45) we obtain $B_{\rm eff} = m\mu$. In the case of a spin unpolarized ground state we get with the help of $\mu = \mu_\uparrow = \mu_\downarrow$ the approximation $B_{\rm eff} = m\mu$ due to the validity of the equations (44), (45). In the case of $\mu_\uparrow \neq \mu_\downarrow$ we obtain the equation $(\mu_\uparrow + \mu_\downarrow)/B_{\rm eff} = 2$ to get the same $q \approx 0$-integrand in the second term of equation (43) using the exact $P_{CS}$ (36) and its approximation (38) in the path integral (20), (39). If we compare this

equation for $\mu_\downarrow = 0$ with (44) and (45) we get a factor two difference. This is because we used for the response functions $\Pi(q, \omega, \mu = 0) = 0$ in equations (44) and (45). This is only correct for $|\omega| > 0$ (see e.g. $\Pi_{a^0 a^0}$ in (A2)). One has to notice that $\Pi(q, \omega, 0)$ is non-zero only on a non-measurable subspace of the $(\omega, q)$-integral of the second term of $\Omega_{5/2}^{ee}$ (43). Thus $\Pi(q, \omega, 0)$ is not relevant for the energy calculation and $B_{\text{eff}} = m\mu$ is a good approximation for $\mu_\downarrow \neq 0$. The absolute value $|\Pi(q, \omega, \mu)|$ of the response functions (with any vertex functions) are growing functions of $\mu$ for $|\omega| > 0$ and zero for $\mu = 0$ and $|\omega| > 0$. Therefore, for a determination of $B_{\text{eff}}$ we only consider that part of the $q \approx 0$ reponse functions of the second term of $\Omega_{5/2}^{ee}$ (43) which contains the larger $\mu$-value $\mu_l$ of $\mu_\uparrow$, $\mu_\downarrow$. The demand that this part of the $q \approx 0$-integrand is identical by using in the path integral either the exact $P_{CS}$ (36) or its approximation (38), results in $B_{\text{eff}} = m\mu_l$. Thus we get for $B_{\text{eff}}$ of the spin polarized as well as the spin unpolarized case the former values.

In the following we calculate the compressibility, the effective mass and the excitations of the spin polarized $\nu = 2 + 1/\tilde{\phi}$ systems. This is done by a restriction of the path integral (20), (39) to one spin component (and inserting $\mu_s = 0$). We now use standard techniques[4] to calculate the compressibility, the effective mass and the excitations of the $\nu = 2 + 1/\tilde{\phi}$ systems in RPA. To calculate the compressibility we have to pay attention to the construction of the path integral of the $\nu = 2 + 1/\tilde{\phi}$ systems. It is then easy to see that in the above model of the spin polarized $\nu = 2 + 1/\tilde{\phi}$ systems the density-density correlation function is given by $\langle|\rho^{5/2}(\vec{r}, l\epsilon)\rho^{5/2}(\vec{r}', l'\epsilon)|\rangle$. In this expression $\rho^{5/2}(\vec{r}, l\epsilon))$ is given by the quantum mechanical Heissenberg operator of the expression $\mathcal{P}_{CS} c_\uparrow^+(\vec{r}) \mathcal{P}_{CS}^+ c_\uparrow(\vec{r})$. The Fourier transformed density-density correlation function could be calculated by analytical continuation of the Fourier transformed sum of $\langle|\sigma_l(\vec{r})\sigma_{l'}(\vec{r}')|\rangle$ and $(V^{ee})^{-1}(\vec{r} - \vec{r}')$. $\langle|\sigma_l(\vec{r})\sigma_{l'}(\vec{r}')|\rangle$ is the $\sigma\sigma$ correlation function of the path integral (20), (39). With the help of the approximations (A4) we obtain for the (retarded) density-density correlation function[32] of the $\nu = 2 + 1/\tilde{\phi}$ system in the range $\omega = 0$ and $q \approx 0$

$$\langle|T\rho^{5/2}(-q, 0)\rho^{5/2}(q, 0)|\rangle_{ret} \quad (46)$$

$$= \frac{\frac{4m^3\mu^2}{(2B_{\text{eff}})^2}\left(1 - \tilde{\phi} + \frac{\tilde{\phi}^2}{4}\right)}{2\pi\left(\frac{4m^3\mu^2}{(2B_{\text{eff}})^2} V^{ee}(q)\left(1 - \tilde{\phi} + \frac{\tilde{\phi}^2}{4}\right) + \left(1 + \frac{\tilde{\phi}^2}{12}\right)\right)} .$$

Here $T$ is the time ordering operator. From (46) we see that $\lim_{q\to 0}\langle|T\rho^{5/2}(-q, 0)\rho^{5/2}(q, 0)|\rangle_{ret}$ is zero for the $\nu = 5/2$ system. By considering higher order terms of the response functions (A2) in the momentum $q$ and $\omega = 0$ we get for the $\nu = 5/2$ system $\langle|T\rho^{5/2}(-q, 0)\rho^{5/2}(q, 0)|\rangle_{ret} = m^2 q^2/(3\pi(2B_{\text{eff}})^2(1 + \tilde{\phi}^2/12)) + O(q^4)$. With the help of the retarded density-density correlation function it is possible two calculate the compressibility $K^{5/2}$ of

the system by using the compressibility sum rule[32] $K^{5/2} = \lim_{q\to 0}(1/\rho^2)\langle|T\rho^{5/2}(-q, 0)\rho^{5/2}(q, 0)|\rangle_{ret}/(1 - (2\pi)V^{ee}(q)\langle|T\rho^{5/2}(-q, 0)\rho^{5/2}(q, 0)|\rangle_{ret})$. Due to this relation we obtain by the definition of $B_{\text{eff}}$ that the compressibility is given by $K^{5/2} = m/(2\pi\rho^2)(1 - \tilde{\phi} + \tilde{\phi}^2/4)/(1 + \tilde{\phi}^2/12)$. Thus we see that the systems of filling fraction $\nu = 2 + 1/\tilde{\phi}$ ($\tilde{\phi} \neq 2$) should be compressible. Furthermore we get the same RPA compressibility as HLR calculated for the $\nu = 1/2$ system except for a factor $(1 - \tilde{\phi} + \tilde{\phi}^2/4)$. In the case of filling fraction $\nu = 5/2$ this is no longer valid. Due to our calculation the $\nu = 5/2$ system is incompressible. Finally, if we take into account all terms of $P$ (36) in the RPA calculation of the compressibility, we get the same expression as (46) for the $\nu = 2 + 1/\tilde{\phi}$ systems. This is due to the definition of $B_{\text{eff}}$ (44).

Next we calculate the effective mass $m^* = m(1 - \partial/(\partial\omega)\Sigma(k_F, 0))/(1 + m/k_F \partial/(\partial k)\Sigma(k_F, 0))$ in RPA. Here $\Sigma(k, \omega)$ is the self energy of the fermions of the path integral (20), (39). Due to the same singular structure of the transversal $(a, a)$ propagator we obtain for the $2 + 1/\tilde{\phi}$ system with the help of (A4) a singular effective mass which is different by a factor $1/(1 - \tilde{\phi} + \tilde{\phi}^2/4)$ from the (diverging) effective mass of the spin polarized $\nu = 1/2$ system calculated by HLR. The expression for the compressibility as well as the effective mass suggests that systems of filling fraction $\nu = 2 + 1/\tilde{\phi}$ behave similar to the $\nu = 1/2$ system except $\tilde{\phi} = 2$. This is in agreement with numerical calculations by Morf and d'Ambrumenil[20].

Like in the $\nu = 1/2$ system one can also derive the cyclotron excitations for the $\nu = 2 + 1/\tilde{\phi}$ systems by a calculation of the density-density correlation function. In this calculation one has to take into account the response functions in the limit (A5).


## V. THE RESULTS OF THE ENERGY CALCULATION

We have calculated the integrals in (31), (43) using numerical methods. The results are shown in figure 2. This figure shows the Coulomb energy as a function of the strength $\alpha$ of the Coulomb interaction function $V^{ee}(k) = e^2/k^\alpha$. It contains the calculation of the spin polarized system , i.e. $\mu_\uparrow = 2\pi\rho/m$, $\mu_\downarrow = 0$, as well as the spin unpolarized system, i.e. $\mu_\uparrow = \mu_\downarrow = \pi\rho/m$, for the $\nu = 1/2$ and the $\nu = 5/2$ system. The graph reveals a transition from a spin singlet to a spin polarized ground state at $\alpha \approx 0.9$ for the $\nu = 1/2$ system. In the $\nu = 5/2$ system the transition is at $\alpha \approx 0.6$. This coincides qualitatively with the numerical result of Jain et al.[23], which has be obtained by exact diagonalization methods on a sphere. Since Jain et al. employed the Haldanes pseudo potentials $V_L$, where the strength of the Coulomb interaction is adjusted on the basis of the ratio of $V_0$ to $V_1$ we can not quantitatively compare the two

results[33]. That is, Jain et al. modify the strength of the pseudo potentials with the help of $\alpha'$ in $[\alpha'V_1, V_1, V_2, ...]$. For the Coulomb interacting $\nu = 1/2$ system the result is $\alpha' = 2.0$. For the Coulomb interacting $\nu = 5/2$ system one gets $\alpha' = 1.45$. Jain et al. obtained a transition from a spin singlet to a spin polarized ground state for the $\nu = 1/2$ system at $\alpha' = 1.1$. They obtained a good overlap of the ground state wave function with a spin polarized Chern-Simons wave function for $\alpha' > 1.1$. The same transition was observed at $\alpha' = 1.2$ for the $\nu = 5/2$ system. Because $\alpha'$ is dominated by the strength of the interaction of neighbouring electrons, the transition at $\alpha < 1$ showing in figure (2) agrees with the results of Jain et al.. But comparing the value $\alpha$ at the transition of the $\nu = 1/2$ system with the $\nu = 5/2$ system, $\alpha$ of the $\nu = 5/2$ system is probably too small. Furthermore, the slope of the two curves in figure 2 of spin $S^2 = \text{Max}$ ($S^2 = |S_z|(|S_z| + 1)$ with $S_z = \rho F/2$) and $S^2 = 0$ of the $\nu = 5/2$ system differs only slightly. Thus the $\nu = 5/2$ system compared to the $\nu = 1/2$ system can change the alpha-value of the transition very easily due to the higher orders of perturbation theory.
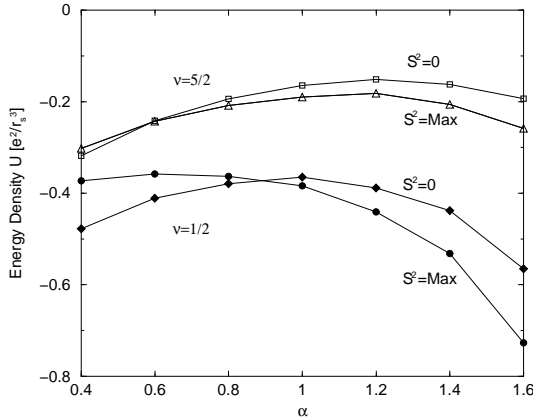


FIG. 2. The Coulomb energy of the ground state of the $\nu = 1/2$ and the $\nu = 5/2$ system including the spin constraints $S^2 = 0$ and $S^2 = \text{Max}$. The Coulomb energy is given in units of $e^2/r_s^3$ where $r_s$ is the effective electron distance $\rho = 1/(\pi r_s)^2$.

Morf shows in[24] using similar numerical diagonalization methods like Jain et al. but considering a larger number of electrons that the ground state of the $\nu = 5/2$ system is not a Chern-Simons wave function for $\alpha = 1$ (Coulomb potential). This ground state wave function shows a large overlap with a spin polarized incompressible paired wave function. For smaller $\alpha'$, respectively $\alpha$, Morf could not identify the ground state wave function. Recently Rezayi and Haldane[25] could identify this ground state as a compressible stripped phase by using similar numerical diagonalization methods like Morf. For larger $\alpha'$, respectively $\alpha$, Morf[24] obtains for the ground state wave function a large overlap with a spin polarized Chern-Simons wave function. Based on this ob-

servation he interpreted the experiments of Eisenstein et al.[22], mentioned in the introduction, that caused on the tilted magnetic field the effective interaction between two neighbouring electrons is increased. Thus the ground state wave function of the system changes to a spin polarized Chern-Simons wave function. At last we mention that Rezayi and Haldane get in[25] a transition of a compressible stripped phase via an incompressible paired quantum Hall state to a Chern-Simons ground state near $\alpha' = 1.45$ even though they considered the pure spin polarized $\nu = 5/2$ system. This shows that the transition is an interaction effect (not caused by the spin degree of freedom).

Comparing for $\alpha = 1$ the Coulomb energies $\Omega_{1/2}^{ee}$, $\Omega_{5/2}^{ee}$ of the ground state (spin polarized) with the Coulomb energies $U_{1/2}^N$, $U_{5/2}^N$ of the numerical calculation method[34] we obtain

$$\Omega_{1/2}^{ee} = -0.13\, e^2\rho^{\frac{3}{2}}, U_{1/2}^N = -0.10\, e^2\rho^{\frac{3}{2}}\,, \qquad (47)$$

$$\Omega_{5/2}^{ee} = -0.067\, e^2\rho^{\frac{3}{2}}, U_{5/2}^N = -0.088\, e^2\rho^{\frac{3}{2}}\,. \qquad (48)$$

Figure 3 shows the Coulomb energy as a function of the variable $S_z/(\rho F)$ for $\alpha = 1$ of the $\nu = 1/2$ and the $\nu = 5/2$ system. $\mu_\uparrow$, $\mu_\downarrow$ are given by $\mu_\downarrow = (1-2S_z/(\rho F))/(1+2S_z/(\rho F))\mu_\uparrow = \pi(1-2S_z/(\rho F))\rho/m$. The total spin $S^2$ is determined by $S^2 = |S_z|(|S_z| + 1)$. The figure shows that the spin polarized ground state, i.e. $S^2 = \text{Max}$, has a minimal Coulomb energy of all $S^2$ ground states. This is in contradiction with recent experimental obeservations[35] which shows that the $\nu = 1/2$ ground state is not fully polarized. Unfortunately these experiments do not show the degree of polarization of the ground state.
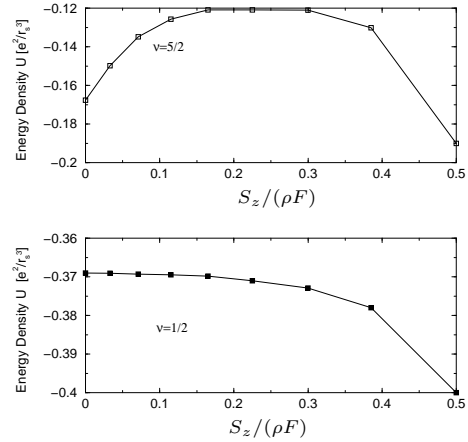


FIG. 3. The Coulomb energy of the $\nu = 1/2$ and the $\nu = 5/2$ system as a function of $S_z/(\rho F)$. The Coulomb energy is given in units of $e^2/r_s^3$ where $r_s$ is the effective electron distance $\rho = 1/(\pi r_s)^2$.

## VI. CONCLUSION

We showed that due to a wrong operator ordering in the path integral of the spin polarized $\nu = 1/2$ system of HLR we got in[15] an IR divergence in the ground state magnetic energy. Therefore we used the well-known Chern-Simons partition function in an operator formulation to get a Chern-Simons path integral with a correct operator ordering. We calculated the energy of this path integral in RPA to show that this path integral has a UV infinite magnetic energy. This UV infinity can be corrected by considering the complete set of Hartree-Fock Feynman graphs of the Chern-Simons Hamiltonian. When calculating the energy of the maximal divergent graphs together with the Hartree-Fock graphs we obtained a well behaved finite magnetic energy. The Coulomb energy is the same as in[15].

Next we developed a formalism to calculate the Coulomb ground state energy of the $\nu = 1/2$ system subjected to a spin $S^2$ constraint and calculated the Coulomb energy as a function of the interaction strength $\sim e^2/k^\alpha$. This was done for a spin polarized ground state as well as for a spin unpolarized ground state. Furthermore, we formulated a Chern-Simons theory of the $\nu = 2 + 1/\tilde{\phi}$ systems by transforming the interaction operator of the electrons from the second to the lowest Landau level getting a $\nu = 1/\tilde{\phi}$ Chern-Simons theory with a modified interaction operator. We used this Chern-Simons theory to calculate the compressibility, the effective mass and the excitations of the spin polarized $\nu = 2 + 1/\tilde{\phi}$ systems in RPA. We see from our calculations that $\nu = 5/2$ is a special system within all $\nu = 2 + 1/\tilde{\phi}$ systems. We get the same compressibility for the $\nu = 2 + 1/\tilde{\phi}$ systems as HLR calculated for the $\nu = 1/2$ system except for a factor $(1 - \tilde{\phi} + \tilde{\phi}^2/4)$. Thus we get that the $\nu = 2 + 1/\tilde{\phi}$ systems are compressible except for $\tilde{\phi} \neq 2$. Due to our RPA calculation the $\nu = 5/2$ is incompressible. Next we calculated the effective mass of the $\nu = 2 + 1/\tilde{\phi}$ systems. For these systems we get a diverging effective mass like HLR got for the $\nu = 1/2$ system with a difference by a factor of $1/(1 - \tilde{\phi} + \tilde{\phi}^2/4)$ ($\tilde{\phi} \neq 2$). The effective mass of the $\nu = 5/2$ system is the same as HLR calculated for the $\nu = 1/2$ system without interaction between the electrons. These calculations (compressibility and effective mass) are in agreement with numerical calculations by Morf and d'Ambrumenil[20] which show that systems of filling fraction $\nu = 2 + 1/\tilde{\phi}$ behave similar to the $\nu = 1/2$ system except for $\tilde{\phi} = 2$. Like in the calculation of the cyclotron excitations of the $\nu = 1/2$ system by HLR we also obtain the cyclotron excitations for the $\nu = 2 + 1/\tilde{\phi}$ systems in RPA.

Next we used the $\nu = 5/2$ Chern-Simons theory to calculate the Coulomb energy as a function of the interaction strength for the spin polarized as well as the spin unpolarized ground state in RPA. We get for the $\nu = 1/2$ as well as the $\nu = 5/2$ system a transition from a spin unpolarized ground state to a polarized ground state at $\alpha < 1$. This agrees with calculations using numerical diagonalization techniques. Furthermore, for $\alpha = 1$ we get a good correspondence to the energies calculated by these numerical methods. At last we calculated the ground state Coulomb energies for the $\nu = 1/2$ and the $\nu = 5/2$ system as a function of the total spin $S^2$ for $\alpha = 1$. We saw that for both systems the maximal spin polarized ground states is actually in the Coulomb energy minimum of all $S^2$ states which is not in agreement with recent experimental observations[35].

Summarizing, already on the level of the RPA, we can see spin effects for both the $\nu = 1/2$ and the $\nu = 5/2$ systems which agree with numerical simulations. As mentioned earlier it is supposed that the Coulomb $\nu = 5/2$ Chern-Simons ground state wave function ($\alpha = 1$) is of a Pfaffian type. Thus the goal still is to develop a theory of the Coulomb interacting $\nu = 5/2$ system which has a mean-field ground state of the paired wave type.

## APPENDIX A: THE RESPONSE FUNCTIONS FOR THE $\nu = 1/2$ AND THE $\nu = 5/2$ SYSTEM

In this section we calculate the response functions for the $\nu = 1/2$ and the $\nu = 5/2$ system. These response functions were used in the energy formulas (9), (31), (43). These response functions are defined by

$$\Pi_{\sigma\sigma}(q, \omega, \mu) \tag{A1}$$
$$= \frac{1}{(2\pi)^3} \int d\Omega \, d^2k \, (\vec{k} + \vec{q})^2 \, \vec{k}^2 \, G(\vec{k} + \vec{q}, \Omega + \omega) \, G(\vec{k}, \Omega) ,$$
$$\Pi_{\sigma a^0}(q, \omega, \mu) = \Pi_{a^0 \sigma}(q, \omega, \mu)$$
$$= \frac{1}{(2\pi)^3} \int d\Omega \, d^2k \, (\vec{k}^2 + \vec{k} \cdot \vec{q}) \, G(\vec{k} + \vec{q}, \Omega + \omega) \, G(\vec{k}, \Omega) ,$$
$$\Pi_{\sigma a}(q, \omega, \mu) = -\Pi_{a\sigma}(q, \omega, \mu)$$
$$= -\frac{i}{(2\pi)^3} \int d\Omega \, d^2k \, \frac{(\vec{k} \times \vec{q})^2}{mq} \, G(\vec{k} + \vec{q}, \Omega + \omega) \, G(\vec{k}, \Omega) ,$$
$$\Pi_{a^0 a^0}(q, \omega, \mu)$$
$$= \frac{1}{(2\pi)^3} \int d\Omega \, d^2k \, G(\vec{k} + \vec{q}, \Omega + \omega) \, G(k, \Omega) ,$$
$$\Pi_{aa}(q, \omega, \mu)$$
$$= \frac{1}{(2\pi)^3} \int d\Omega \, d^2k \, \frac{(\vec{k} \times \vec{q})^2}{m^2 q^2} \, G(\vec{k} + \vec{q}, \Omega + \omega) \, G(\vec{k}, \Omega) .$$

G is the Green's function $G(\vec{q}, \omega) = -1/(i\omega - q^2/(2m) + \mu)$. After performing the integrals in (A1) one gets with the help of the substitution $u = \frac{q^2}{2m}$

$$\Pi^1_{\sigma\sigma}(q,\omega,\mu) = \frac{m^3}{4\pi} \sum_{\sigma\in\{-1,1\}} \left[ + \frac{4}{30u^3}(u+i\sigma\omega)^5 \right. \qquad (A2)$$

$$-\frac{1}{15u^3}\sqrt{(u+i\sigma\omega)^2-4u\mu}$$

$$\left. \times \left(2(u+i\sigma\omega)^4 + 4(u+i\sigma\omega)^2u\mu + 12\,u^2\mu^2\right)\right] ,$$

$$\Pi^2_{\sigma\sigma}(q,\omega,\mu) = \frac{m^3}{4\pi} \sum_{\sigma\in\{-1,1\}} \left[ 2\frac{(i\sigma\omega)}{3u^2}\left\{(u+i\sigma\omega)^3 \right.\right.$$

$$\left.\left. -\sqrt{(u+i\sigma\omega)^2-4u\mu}\left((u+i\sigma\omega)^2+2u\mu\right)\right\} - 4\mu^2 \right] ,$$

$$\Pi^1_{\sigma a^0}(q,\omega,\mu) = \frac{m^2}{4\pi} \sum_{\sigma\in\{-1,1\}} \left[ -\frac{1}{3u^2}(u+i\sigma\omega)^3 \right.$$

$$\left. +\frac{1}{3u^2}\sqrt{(u+i\sigma\omega)^2-4u\mu}\left((u+i\sigma\omega)^2+2u\mu\right)\right] ,$$

$$\Pi^2_{\sigma a^0}(q,\omega,\mu) = \frac{m^2}{4\pi} \sum_{\sigma\in\{-1,1\}} \left[ \frac{(u+i\sigma\omega)}{u} \right.$$

$$\left. \times \left((u+i\sigma\omega) - \sqrt{(u+i\sigma\omega)^2-4u\mu}\right) - 2\mu \right] ,$$

$$\Pi_{\sigma a}(q,\omega,\mu) = -i\,m^{\frac{3}{2}}\sqrt{2u}\,\Pi_{aa}(q,\omega,\mu) ,$$

$$\Pi_{aa}(q,\omega,\mu) = \frac{1}{2\pi} \sum_{\sigma\in\{-1,1\}} \left[ \frac{1}{12u^2}\left\{(u+i\sigma\omega)^3 \right.\right.$$

$$\left.\left. -\sqrt{\left((u+i\sigma\omega)^2-4u\mu\right)^3}\right\} - \frac{\mu}{2}\right],$$

$$\Pi_{a^0 a^0}(q,\omega,\mu) = \frac{m}{2\pi} \sum_{\sigma\in\{-1,1\}} \left[ \frac{1}{2u}\sqrt{(u+i\sigma\omega)^2-4u\mu} - \frac{1}{2}\right].$$

$\Pi_{\sigma\sigma}$ and $\Pi_{\sigma a^0}$ are given by

$$\Pi_{\sigma\sigma} = \Pi^1_{\sigma\sigma} + \Pi^2_{\sigma\sigma}, \quad , \qquad \Pi_{\sigma a^0} = \Pi^1_{\sigma a^0} + \Pi^2_{\sigma a^0} . \quad (A3)$$

For calculating the compressibility, the effective mass and the excitations we have to continue analytically the response functions $\Pi$ to imaginary frequencies[4]. In the range $\omega \ll \frac{q^2}{2m}$ and $q \ll k_F$ we get

$$\Pi_{\sigma\sigma}(q,-i\omega,\mu) = -\frac{4\,m^3\mu^2}{2\pi}, \Pi_{\sigma a^0}(q,-i\omega,\mu) = -\frac{2\,m^2\mu}{2\pi},$$

$$\Pi_{\sigma a}(q,-i\omega,\mu) = i\,q\frac{m\,\mu}{2\pi}, \qquad \Pi_{a^0 a^0}(q,-i\omega,\mu) = -\frac{m}{2\pi},$$

$$\Pi_{aa}(q,-i\omega,\mu) = \frac{q^2}{24\pi m} - \frac{\mu}{2\pi} - i\sqrt{2m\mu}\frac{\omega}{2\pi q}. \quad (A4)$$

By multiplying every value in (A4) by $\omega/(qk_F)$, one can calculate the asymptotic of the next term in the expansion of the response functions. Comparing $\Pi_{aa}$ in (A4) with the corresponding term of HLR[4], the result differs by a factor two in the first term of $\Pi_{aa}$. This difference is also observed in the case of the $\nu = 1/2$ Chern-Simons system with impurities[36].

Furthermore, we need the analytical continued response functions in the range $\frac{q^2}{2m} \ll \omega$ and $q \ll k_F$ for a calculation of the cyclotron excitations . In this range we get

$$\Pi_{\sigma\sigma}(q,-i\omega,\mu) = O\left(\frac{q^2}{m\omega}\right), \Pi_{\sigma a^0}(q,-i\omega,\mu) = O\left(\frac{q^2}{m\omega}\right),$$

$$\Pi_{\sigma a}(q,-i\omega,\mu) = O\left(\frac{q^2}{m\omega}\right), \quad \Pi_{a^0 a^0}(q,-i\omega,\mu) = \frac{\mu}{2\pi}\frac{q^2}{\omega^2},$$

$$\Pi_{aa}(q,-i\omega,\mu) + \frac{\mu}{2\pi} = \frac{\mu}{2\pi} . \quad (A5)$$

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